Education & Science Journal of
Policy Review and Curriculum Development
Hard Print: 2315-8425 Online: 2354-1660

November, 2016

Vol. 6, No. 1

Diagonally Implicit Super Class of Block Backward Differentiation Formula with Off-Step Points for Solving Stiff Initial Value Problems

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Abstract

A new formula called 2-point diagonally implicit super class of BBDF with two off-step points (2ODISBBDF) for solving stiff IVPs is formulated. The method approximates two solutions with two off-step points simultaneously at each iteration. By varying a parameter p E $(-1,1)$ in the formula, different sets of formulae can be generated. A specific choice of $p = \frac{3}{4}$ is made and it was shown that the method is both zero and A-stable. A comparison between the new method and the existing 2-point block backward differentiation formula with off-step points (2OBBDF) is made. The results show that the new method outperformed existing 2OBBDF method in terms of accuracy.

Keywords:
Off-step, Diagonally Implicit Super Class of Block Backward Differentiation Formula, Stiff IVPs, Implicit Block Method, A-stability.

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Background to the Study

Consider a system of first order stiff initial value problems (IVPs) of the form:

$$y_i' = f_i(x, \tilde{Y}), \qquad i = 1,\ 2, ..., n. \qquad (1)$$

With $\tilde{Y}(a) = \eta$, in the interval $a \le x \le b$, where $\tilde{Y} = (y_1, y_1',\ y_2' ..., y_n')$ and $\tilde{\eta} = (\eta_1',\ \eta_2' ..., \eta_n')$.

System (1) is said to be stiff if it contains widely varying time scales, i.e., some components of the solution decay much more rapidly than others. Most realistic stiff systems do not have analytical solutions so that a numerical procedure must be used. Stiff ODEs occur in many fields of engineering and physical sciences such as electrical circuits, vibrations, chemical reactions, kinetics etc.

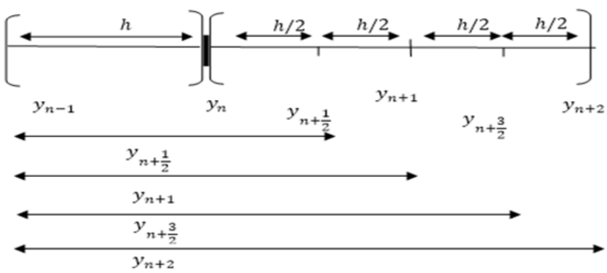
Developing methods for solving stiff problems remains a challenge in modern numerical analysis. Curtiss and Hirschfelder (1952) discover Backward Differentiation Formula (BDF). Since then most of the improvements in the class of linear multistep methods have been based on BDF because of its special properties. Ibrahim (2007) introduced r-point block BDF (BBDF). Super class of block BDF, which is both zero and A-stable, was developed by Suleiman (2014). The method is derived from 2-point block BDF and outperformed both 2BBDF and 1BDF in terms of accuracy.

In order to gain an efficient numerical approximation in terms of accuracy and computational time, a super class of diagonally implicit BBDF method can be considered. The study of diagonally implicit for multistep attracted some researchers such Ababneh (2009), Alexander (1977), Musa (2016) and Zawawi (2012). Abasi (2014) developed a 2-point Block BDF Method with off-step points for solving Stiff ODEs which differs from all the methods above because it calculates two solution values with off-step points simultaneously at each iteration. The motivation of this research is to develop a new method that would be called diagonally implicit super class of BBDF with off-step points

Derivation

In this work, two solution values, y_{n+1} and y_{n+2} and two off-step points $y_{n+\frac{1}{2}}$ and $y_{n+\frac{3}{2}}$ which are chosen at the points where the step size is halved, are formulated in a block simultaneously. The formulae are computed using two back values y_{n-1} and y_n with step size h. The formula is derived with the aid of this diagram below:

Figure 1: Points involved in 2-Point Super Class BBDF with off-step points method.



Definition 2.1: the 2-point super class of block backward differentiation formula with off-step points is defined as

$$\sum_{j=0}^{1+k} \alpha_{j,i} y_{n+j-1} = h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-\frac{1}{2}}\right), \qquad k = i = \frac{1}{2}, 1, \frac{3}{2}, 2. \qquad (2)$$

Where $k = i = \frac{1}{2}$ represents the first point, $k = I = 1$ represents the second point, $k = i = \frac{3}{2}$ represents the third point and $k = I = 1$ represents the fourth point. The formula (2) is derived from Taylor's series expansion as follows:

Definition 2.2: Linear operator L_i associated with first, second, third and fourth point DI2SBBDF with off-step point is defined by

$$L_i[y(x_n), h]: \alpha_{0,i} y_{n-1} + \alpha_{1,i} y_n + \alpha_{\frac{3}{2},i} y_{n+\frac{1}{2}} - h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-\frac{1}{2}}\right) = 0, \quad k = i = \frac{1}{2}. \quad (3)$$

$$L_i[y(x_n), h]: \alpha_{0,i} y_{n-1} + \alpha_{1,i} y_n + \alpha_{\frac{3}{2},i} y_{n+\frac{1}{2}} + \alpha_{2,i} y_{n+1} - h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-\frac{1}{2}}\right) = 0, \quad k = i = 1. \quad (4)$$

$$L_i[y(x_n), h]: \alpha_{0,i} y_{n-1} + \alpha_{1,i} y_n + \alpha_{\frac{3}{2},i} y_{n+\frac{1}{2}} + \alpha_{2,i} y_{n+1} + \alpha_{\frac{5}{2},i} y_{n+\frac{3}{2}} - h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-\frac{1}{2}}\right) = 0, \quad k = i = \frac{3}{2}. \quad (5)$$

$$L_i[y(x_n), h]: \alpha_{0,i} y_{n-1} + \alpha_{1,i} y_n + \alpha_{\frac{3}{2},i} y_{n+\frac{1}{2}} + \alpha_{2,i} y_{n+1} + \alpha_{\frac{5}{2},i} y_{n+\frac{3}{2}} + \alpha_{3,i} y_{n+2} - h\beta_{k,i}\left(f_{n+k} - \rho f_{n+k-\frac{1}{2}}\right) = 0, \quad k = i = 2. \quad (6)$$

First point: To derive the first point $y_{n+\frac{1}{2}}$, $k = I = \frac{1}{2}$ and define the operator as

$$\alpha_{0,\frac{1}{2}} y_{n-1} + \alpha_{1,\frac{1}{2}} y_n + \alpha_{\frac{3}{2},\frac{1}{2}} y_{n+\frac{1}{2}} - h\beta_{\frac{1}{2},\frac{1}{2}}\left(f_{n+\frac{1}{2}} - \rho f_n\right). \quad (7)$$

Expanding (7) as Taylor series about x_n and collecting like terms gives

$$C_{0,\frac{1}{2}} y(x_n) + C_{1,\frac{1}{2}} h y'(x_n) + C_{\frac{3}{2},\frac{1}{2}} h^2 y''(x_n) + \cdots \quad (8)$$

Where

$$C_{0,\frac{1}{2}} = \alpha_{0,\frac{1}{2}} + \alpha_{1,\frac{1}{2}} + \alpha_{\frac{3}{2},\frac{1}{2}} = 0,$$

$$C_{1,\frac{1}{2}} = -\alpha_{0,\frac{1}{2}} + \frac{1}{2}\alpha_{\frac{3}{2},\frac{1}{2}} + \beta_{\frac{1}{2},\frac{1}{2}}(\rho - 1) = 0, \quad (9)$$

$$C_{\frac{3}{2},\frac{1}{2}} = \frac{1}{2}\alpha_{0,\frac{1}{2}} + \frac{1}{8}\alpha_{\frac{3}{2},\frac{1}{2}} - \frac{1}{2}\beta_{\frac{1}{2},\frac{1}{2}} = 0.$$

The coefficient of $y_{n+\frac{1}{2}}, \alpha_{\frac{3}{2}\frac{1}{2}}$ is normalized to 1. Solving the simultaneously equation (9) for the values of $\alpha_{j,1}'s$ and $\beta_{j,1}'s$ gives the formula for $y_{n+\frac{1}{2}}$ as

$$y_{n+\frac{1}{2}} = \frac{1}{4}\frac{\rho+1}{\rho-2}y_{n-1} + \frac{3}{4}\frac{\rho-3}{\rho-2}y_n + \frac{3}{4(\rho-2)}\rho h f_n - \frac{3}{4(\rho-2)}hf_{n+\frac{1}{2}} \, . \qquad (10)$$

Similar procedure is applied as in the derivation of first point to obtain the second, third and fourth points as

$$y_{n+1} = -\frac{1}{3}\frac{\rho+1}{3\rho-14}y_{n-1} + \frac{2(3\rho+4)}{3\rho-14}y_n - \frac{8}{3}\frac{\rho+8}{3\rho-14}y_{n+\frac{1}{2}} + \frac{4}{3\rho-14}\rho hf_{n+\frac{1}{2}} - \frac{4}{3\rho-14}f_{n+1}. \qquad (11)$$

$$y_{n+\frac{3}{2}} = \frac{1}{2}\frac{\rho+3}{8\rho-61}y_{n-1} - \frac{5(\rho+5)}{8\rho-61}y_n + \frac{5(8\rho+15)}{8\rho-61}y_{n+\frac{1}{2}} - \frac{45}{2}\frac{\rho+5}{8\rho-61}y_{n+1} + \frac{15}{8\rho-61}\rho hf_{n+1} - \frac{15}{8\rho-61}hf_{n+\frac{3}{2}}. \qquad (12)$$

$$y_{n+2} = -\frac{1}{5}\frac{4+\rho}{-54+5\rho}y_{n-1} + \frac{18+5\rho}{-54+5\rho}y_n - \frac{4(16+5\rho)}{-54+5\rho}y_{n+\frac{1}{2}} + \frac{9(12+5\rho)}{-54+5\rho}y_{n+1} - \frac{4}{5}\frac{144+31\rho}{-54+5\rho}y_{n+\frac{3}{2}} + \frac{12}{-54+5\rho}\rho hf_{n+\frac{3}{2}} - \frac{12}{-54+5\rho}hf_{n+2} \, . \qquad (13)$$

For absolute stability of the method, ρ is Chosen to be in the interval $(-1, 1)$ as in Suleiman (2014). By choosing $\rho = \frac{3}{4}$ in equation (10), (11), (12) and (13) to obtain the 2-point diagonally implicit super class of BBDF with off-step points as follows:

$$\begin{aligned}
y_{n+\frac{1}{2}} &= -\frac{7}{20}y_{n-1} + \frac{27}{20}y_n - \frac{9}{20}hf_n + \frac{3}{5}hf_{n+\frac{1}{2}} \; , \\
y_{n+1} &= \frac{11}{141}y_{n-1} - \frac{50}{47}y_n + \frac{280}{141}y_{n+\frac{1}{2}} - \frac{12}{47}hf_{n+\frac{1}{2}} + \frac{16}{47}hf_{n+1} \; , \\
y_{n+\frac{3}{2}} &= -\frac{3}{88}y_{n-1} + \frac{13}{22}y_n - \frac{21}{11}y_{n+\frac{1}{2}} + \frac{207}{88}y_{n+1} - \frac{9}{44}hf_{n+1} + \frac{3}{11}hf_{n+\frac{3}{2}} \; , \qquad (14) \\
y_{n+2} &= \frac{19}{1005}y_{n-1} - \frac{29}{67}y_n + \frac{316}{201}y_{n+\frac{1}{2}} - \frac{189}{67}y_{n+1} + \frac{892}{335}y_{n+\frac{3}{2}} - \frac{12}{67}hf_{n+\frac{3}{2}} + \frac{16}{67}hf_{n+2} \; .
\end{aligned}$$

In matrix form, equation (14) can be written as

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{280}{141} & 1 & 0 & 0 \\ \frac{21}{11} & -\frac{207}{88} & 1 & 0 \\ -\frac{316}{201} & \frac{189}{67} & -\frac{892}{335} & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix}
= \begin{pmatrix} 0 & -\frac{7}{20} & 0 & \frac{27}{20} \\ 0 & \frac{11}{141} & 0 & -\frac{50}{47} \\ 0 & -\frac{3}{88} & 0 & \frac{13}{22} \\ 0 & \frac{19}{1005} & 0 & -\frac{29}{67} \end{pmatrix} \begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & -\frac{9}{20} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix}
+ h \begin{pmatrix} \frac{3}{5} & 0 & 0 & 0 \\ -\frac{12}{47} & \frac{16}{47} & 0 & 0 \\ 0 & -\frac{9}{44} & \frac{3}{11} & 0 \\ 0 & 0 & -\frac{12}{67} & \frac{16}{67} \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix}. \qquad (15)
$$

Definition 2.3: Method (15) is diagonally implicit if the matrix in its left hand side is an upper triangular.

Order of the Method

This section derives the order of the method corresponding to the equations in (14). It can be written in the following form

$$
\begin{aligned}
&y_{n+\frac{1}{2}} + \frac{7}{20} y_{n-1} - \frac{27}{20} y_n = -\frac{9}{20} h f_n + \frac{3}{5} h f_{n+\frac{1}{2}} \ , \\
&y_{n+1} - \frac{11}{141} y_{n-1} + \frac{50}{47} y_n - \frac{280}{141} y_{n+\frac{1}{2}} = -\frac{12}{47} h f_{n+\frac{1}{2}} + \frac{16}{47} h f_{n+1} \ , \\
&y_{n+\frac{3}{2}} + \frac{3}{88} y_{n-1} - \frac{13}{22} y_n + \frac{21}{11} y_{n+\frac{1}{2}} - \frac{207}{88} y_{n+1} = -\frac{9}{44} h f_{n+1} + \frac{3}{11} h f_{n+\frac{3}{2}} \ , \\
&y_{n+2} - \frac{19}{1005} y_{n-1} + \frac{29}{67} y_n - \frac{316}{201} y_{n+\frac{1}{2}} + \frac{189}{67} y_{n+1} - \frac{892}{335} y_{n+\frac{3}{2}} = -\frac{12}{67} h f_{n+\frac{3}{2}} + \frac{16}{67} h f_{n+2}.
\end{aligned} \qquad (16)
$$

Equation (16) can be written as in matrix form as

$$\begin{pmatrix} 0 & \frac{7}{20} & 0 & -\frac{27}{20} \\ 0 & -\frac{11}{141} & 0 & \frac{50}{47} \\ 0 & \frac{3}{88} & 0 & -\frac{13}{22} \\ 0 & -\frac{19}{1005} & 0 & \frac{29}{67} \end{pmatrix} \begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{280}{141} & 1 & 0 & 0 \\ \frac{21}{11} & -\frac{207}{88} & 1 & 0 \\ -\frac{316}{201} & \frac{189}{67} & -\frac{892}{335} & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix}$$

$$= h \begin{pmatrix} 0 & 0 & 0 & -\frac{9}{20} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix}$$

$$+ h \begin{pmatrix} \frac{3}{5} & 0 & 0 & 0 \\ -\frac{12}{47} & \frac{16}{47} & 0 & 0 \\ 0 & -\frac{9}{44} & \frac{3}{11} & 0 \\ 0 & 0 & -\frac{12}{67} & \frac{16}{67} \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix}. \qquad (17)$$

Le $D_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, D_1 = \begin{pmatrix} \frac{7}{20} \\ -\frac{11}{141} \\ \frac{3}{88} \\ -\frac{19}{1005} \end{pmatrix}, D_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, D_3 = \begin{pmatrix} -\frac{27}{20} \\ \frac{50}{47} \\ -\frac{13}{22} \\ \frac{29}{67} \end{pmatrix}, D_4 = \begin{pmatrix} 1 \\ -\frac{280}{141} \\ \frac{21}{11} \\ -\frac{316}{201} \end{pmatrix}, D_5 = \begin{pmatrix} 0 \\ 1 \\ -\frac{207}{88} \\ \frac{189}{67} \end{pmatrix}, D_6 =$
$\begin{pmatrix} 0 \\ 0 \\ 1 \\ -\frac{892}{335} \end{pmatrix}, D_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, G_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, G_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, G_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, G_3 = \begin{pmatrix} -\frac{9}{20} \\ 0 \\ 0 \\ 0 \end{pmatrix}, G_4 = \begin{pmatrix} \frac{3}{5} \\ \frac{12}{47} \\ 0 \\ 0 \end{pmatrix}, G_5 =$
$\begin{pmatrix} 0 \\ \frac{16}{47} \\ -\frac{9}{44} \\ 0 \end{pmatrix}, G_6 = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{11} \\ -\frac{12}{67} \end{pmatrix}$ tand $G_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{16}{67} \end{pmatrix}$.

Definition 3.1: The order of the block method (14) and its associated linear operator L given by

$$L[y(x); h] = \sum_{j=0}^{7} \left[D_j y\left(x + j\frac{h}{2}\right) - hG_j y'\left(x + j\frac{h}{2}\right) \right], (18)$$

Expanding the function $y\left(x + j\frac{h}{2}\right)$ and its derivative $y'\left(x + j\frac{h}{2}\right)$ as Taylor series around x gives

$$y\left(x + j\frac{h}{2}\right) = y(x) + j\frac{h}{2}y'(x) + \frac{(j\frac{h}{2})^2}{2!}y''(x) + \frac{(j\frac{h}{2})^3}{3!}y'''(x) + \cdots (19)$$

$$y'\left(x + j\frac{h}{2}\right) = y'(x) + j\frac{h}{2}y''(x) + \frac{(j\frac{h}{2})^2}{2!}y'''(x) + \frac{(j\frac{h}{2})^3}{3!}y^{iv}(x) + \cdots (20)$$

Substituting (19) and (20) into (18) represents

$$L[y(x);h] = \sum_{j=0}^{7} D_j[y(x) + j\frac{h}{2}y'(x) + \frac{\left(j\frac{h}{2}\right)^2}{2!}y''(x) + \frac{\left(j\frac{h}{2}\right)^3}{3!}y'''(x)\ldots] - h\sum_{j=0}^{7} G_j[y'(x) + j\frac{h}{2}y''(x)$$

$$+ \frac{(j\frac{h}{2})^2}{2!}y'''(x) + \frac{(j\frac{h}{2})^3}{3!}y^{iv}(x) + \cdots]$$

$$= \sum_{j=0}^{7} D_j[y(x)] + \frac{1}{2}\sum_{j=0}^{k}[jD_j - 2G_j]hy'(x) + \frac{1}{4}\sum_{j=0}^{k}\left[\frac{1}{2!}j^2D_j - 2jG_j\right]h^2y''(x)$$

$$+ \frac{1}{8}\sum_{j=0}^{k}\left[\frac{1}{3!}j^3D_j - 2\frac{1}{2!}j^2G_j\right]h^3y'''(x) + \cdots \qquad (21)$$

The difference operator (21) and the associated method (14) is considered of order p if $E_0 = E_1 = E_2 = \cdots E_p = 0$ and $E_{p+1} \neq 0$

In this case

$$E_0 = \sum_{j=0}^{7} D_j = 0 \quad . \qquad (22)$$

$$E_1 = \sum_{j=0}^{7} (jD_j - 2G_j) = 0 \quad . \qquad (23)$$

$$E_2 = \sum_{j=0}^{7} (\frac{1}{2!}j^2D_j - 2jG_j) = 0 \quad . \qquad (24)$$

$$E_3 = \sum_{j=0}^{7} (\frac{1}{3!}j^3D_j - 2\frac{1}{2!}j^2G_j) = \begin{pmatrix} -\frac{9}{10} \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (25)$$

Therefore, the method (14) is of order 2, with error constant $E_3 = \begin{pmatrix} -\frac{9}{10} \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

The stability properties of method (14) are discussed here. We begin by defining zero and A-stability taken from Suleiman (2014)

Definition 4.1: A LMM is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple.

Definition 4.2: A LMM is said to be A-stable if its stability region covers the entire negative half-plane

The method (14) can be rewritten in matrix form as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{280}{141} & 1 & 0 & 0 \\ \frac{21}{11} & -\frac{207}{88} & 1 & 0 \\ -\frac{316}{201} & \frac{189}{67} & -\frac{892}{335} & 1 \end{pmatrix}\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{7}{20} & 0 & \frac{27}{20} \\ 0 & \frac{11}{141} & 0 & -\frac{50}{47} \\ 0 & -\frac{3}{88} & 0 & \frac{13}{22} \\ 0 & \frac{19}{1005} & 0 & -\frac{29}{67} \end{pmatrix}\begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -\frac{9}{20} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} + h\begin{pmatrix} \frac{3}{5} & 0 & 0 & 0 \\ -\frac{12}{47} & \frac{16}{47} & 0 & 0 \\ 0 & -\frac{9}{44} & \frac{3}{11} & 0 \\ 0 & 0 & -\frac{12}{67} & \frac{16}{67} \end{pmatrix}\begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix}. \qquad (26)$$

Definition 4.3: Let Y_m and F_m be vectors defined by

$Y_m = [y_{n+1}, y_{n+2}, \dots, y_{n+r}]^T, F_m = [f_{n+1}, f_{n+2}, \dots, f_{n+r}]^T r = 2,\ and\ n = 2m$(see [8]).

Method (14) can be written in matrix form as follows:

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m)\ . \qquad (27)$$

where $A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{280}{141} & 1 & 0 & 0 \\ \frac{21}{11} & -\frac{207}{88} & 1 & 0 \\ -\frac{316}{201} & \frac{189}{67} & -\frac{892}{335} & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & -\frac{7}{20} & 0 & \frac{27}{20} \\ 0 & \frac{11}{141} & 0 & -\frac{50}{47} \\ 0 & -\frac{3}{88} & 0 & \frac{13}{22} \\ 0 & \frac{19}{1005} & 0 & -\frac{29}{67} \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 0 & 0 & -\frac{9}{20} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} \frac{3}{5} & 0 & 0 & 0 \\ -\frac{12}{47} & \frac{16}{47} & 0 & 0 \\ 0 & -\frac{9}{44} & \frac{3}{11} & 0 \\ 0 & 0 & -\frac{12}{67} & \frac{16}{67} \end{pmatrix}, Y_{m-1} = \begin{pmatrix} y_{n-\frac{3}{2}} \\ y_{n-1} \\ y_{n-\frac{1}{2}} \\ y_n \end{pmatrix} = \begin{pmatrix} y_{2m-\frac{3}{2}} \\ y_{2m-1} \\ y_{2m-\frac{1}{2}} \\ y_{2m} \end{pmatrix} = \begin{pmatrix} y_{2(m-1)+\frac{1}{2}} \\ y_{2(m-1)+1} \\ y_{2(m-1)+\frac{3}{2}} \\ y_{2(m-1)+2} \end{pmatrix}, Y_m = \begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} y_{2m+\frac{1}{2}} \\ y_{2m+1} \\ y_{2m+\frac{3}{2}} \\ y_{2m+2} \end{pmatrix}, F_{m-1} = \begin{pmatrix} f_{n-\frac{3}{2}} \\ f_{n-1} \\ f_{n-\frac{1}{2}} \\ f_n \end{pmatrix} = \begin{pmatrix} f_{2m-\frac{3}{2}} \\ f_{2m-1} \\ f_{2m-\frac{1}{2}} \\ f_{2m} \end{pmatrix} = \begin{pmatrix} f_{2(m-1)+\frac{1}{2}} \\ f_{2(m-1)+1} \\ f_{2(m-1)+\frac{3}{2}} \\ f_{2(m-1)+2} \end{pmatrix}, F_m = \begin{pmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{pmatrix} = \begin{pmatrix} f_{2m+\frac{1}{2}} \\ f_{2m+1} \\ f_{2m+\frac{3}{2}} \\ f_{2m+2} \end{pmatrix}.$

Substituting scalar test equation $y' = \lambda y$ ($\lambda < 0$, λ complex) into (27) and using $\lambda h = \bar{h}$gives

$$A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 Y_{m-1} + B_1 Y_m). \qquad (28)$$

The stability polynomial of (14) is given by

$$\mathrm{Det}\left[(A_0 - \bar{h}B_1)t - (A_1 + \bar{h}B_0)\right] = 0. \qquad (29)$$

i.e

$$R(t,\bar{h}) = t^4 - \frac{24874}{18425}t^3 + \frac{6449}{18425}t^2 + \frac{49463}{346390}t^2\bar{h} + \frac{10734}{865975}\bar{h}^2t^2 + \frac{1447}{157450}\bar{h}t^3 - \frac{4304}{15745}\bar{h}^2t^3 + \frac{8283}{157450}\bar{h}^3t^3 - \frac{4304}{15745}\bar{h}^3t^2 - \frac{729}{173195}\bar{h}^4t^3 - \frac{251472}{173195}\bar{h}t^4 + \frac{129973}{173195}\bar{h}^2t^4 - \frac{28704}{173195}\bar{h}^3t^4 + \frac{2304}{173195}\bar{h}^4t^4 = 0. \qquad (30)$$

For zero stability, we set $\bar{h} = 0$ in (30) to obtain

$$t^4 - \frac{24874}{18425}t^3 + \frac{6449}{18425}t^2 = 0. \qquad (31)$$

Solving equation (31) for t gives the following roots:

$t = 0, t = 0, t = 0.350014$ and $t = 1$.(32)

From the definition 4.1, method (14) is zero-stable. The stability region of method (14) is determined by substituting $t = e^{i\theta}$ and the graph is shown below:

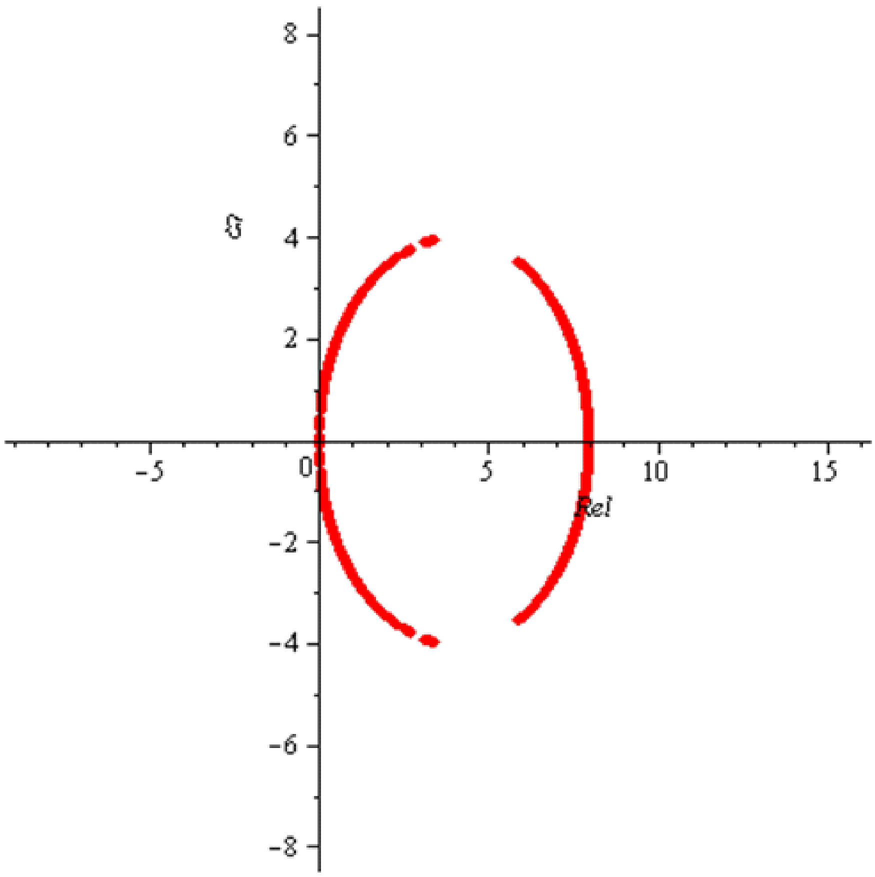


Figure 2: Stability Region of the 2-Point Super Class BBDF with off-step points.
From the definition 4.2, method (14) is A-stable

Implementation of the Method

Newton's iteration is used in implementing the method. The procedure is described as follows. We begin by defining the error.

Definition 5.1

Let y_i *and* $y(x_i)$be the approximate solution of (1)

Then the absolute error is given by

$$(error_i)_t = |(y_i)_t - (y(x_i))_t|. \qquad (33)$$

The maximum error is defined by

$$\text{MAXE} = \underbrace{max}_{1 \le i \le T}\left(\underbrace{\max(error_i)_t}_{1 \le i \le N}\right), \quad (34)$$

where T is the total number of steps and N is the number of equations.

Define from (14)

$$\begin{aligned}
F_{\frac{1}{2}} &= y_{n+\frac{1}{2}} + \frac{9}{20}hf_n - \frac{3}{5}hf_{n+\frac{1}{2}} + \varepsilon_{\frac{1}{2}} \ , \\
F_1 &= y_{n+1} - \frac{280}{141}y_{n+\frac{1}{2}} + \frac{12}{47}hf_{n+\frac{1}{2}} - \frac{16}{47}hf_{n+1} + \varepsilon_1 \ , \\
F_{\frac{3}{2}} &= y_{n+\frac{3}{2}} + \frac{21}{11}y_{n+\frac{1}{2}} - \frac{207}{88}y_{n+1} + \frac{9}{44}hf_{n+1} - \frac{3}{11}hf_{n+\frac{3}{2}} + \varepsilon_{\frac{3}{2}} \ , \qquad (35) \\
F_2 &= y_{n+2} - \frac{316}{201}y_{n+\frac{1}{2}} + \frac{189}{67}y_{n+1} - \frac{892}{335}y_{n+\frac{3}{2}} + \frac{12}{67}hf_{n+\frac{3}{2}} = \frac{16}{67}hf_{n+2} + \varepsilon_2.
\end{aligned}$$

Where $\varepsilon_{\frac{1}{2}} = \frac{7}{20}y_{n-1} - \frac{27}{20}y_n$, $\varepsilon_1 = -\frac{11}{141}y_{n-1} + \frac{50}{47}y_n$, $\varepsilon_{\frac{3}{2}} = \frac{3}{88}y_{n-1} - \frac{13}{22}y_n$ and

$\varepsilon_2 = -\frac{19}{1005}y_{n-1} + \frac{29}{67}y_n$ are the back values.

Let $y_{n+j}^{(i+1)}$, $j = \frac{1}{2}, 1, \frac{3}{2}, 2$, denote the (i + 1)th iterative values of y_{n+j} and define

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+1)} - y_{n+j}^{(i)}, j = \frac{1}{2}, 1, \frac{3}{2}, 2. \qquad (36)$$

Newton's iteration for the 2-point SBBDF with off-step point method takes the form:

$$e_{n+j}^{(i+1)} = -[F_j'(y_{n+j}^{(i)})]^{-1}[F_j(y_{n+j}^{(i)})], \ j=\frac{1}{2}, 1, \frac{3}{2}, 2. \qquad (37)$$

This can be written as

$$\left[F_j'\left(y_{n+j}^{(i)}\right)\right] e_{n+j}^{(i+1)} = -[F_j(y_{n+j}^{(i)})] \qquad j = \frac{1}{2}, 1, \frac{3}{2}, 2. \quad (38)$$

and in matrix form, equation (38) is equivalent to

$$
\begin{pmatrix}
1-h\frac{3}{5}\frac{\partial f^{(i)}_{n+\frac{1}{2}}}{\partial y^{(i)}_{n+\frac{1}{2}}} & 0 & 0 & 0\\
-\frac{280}{141}+\frac{12}{47}\frac{\partial f^{(i)}_{n+1}}{\partial y^{(i)}_{n+\frac{1}{2}}} & 1-\frac{16}{47}h\frac{\partial f^{(i)}_{n+1}}{\partial y^{(i)}_{n+1}} & 0 & 0\\
\frac{21}{11} & -\frac{207}{88}+\frac{9}{44}h\frac{\partial f^{(i)}_{n+\frac{3}{2}}}{\partial y^{(i)}_{n+1}} & 1-\frac{3}{11}h\frac{\partial f^{(i)}_{n+\frac{3}{2}}}{\partial y^{(i)}_{n+\frac{3}{2}}} & 0\\
-\frac{316}{201} & \frac{189}{67} & -\frac{892}{335}+\frac{12}{67}h\frac{\partial f^{(i)}_{n+2}}{\partial y^{(i)}_{n+\frac{3}{2}}} & 1-\frac{16}{67}h\frac{\partial f^{(i)}_{n+2}}{\partial y^{(i)}_{n+2}}
\end{pmatrix}
\begin{pmatrix}
e^{(i+1)}_{n+\frac{1}{2}}\\ e^{(i+1)}_{n+1}\\ e^{(i+1)}_{n+\frac{3}{2}}\\ e^{(i+1)}_{n+2}
\end{pmatrix}
=
$$

$$
\begin{pmatrix}
-1 & 0 & 0 & 0\\
\frac{280}{141} & -1 & 0 & 0\\
-\frac{21}{11} & \frac{207}{88} & -1 & 0\\
\frac{316}{201} & -\frac{189}{67} & -\frac{892}{335} & -1
\end{pmatrix}
\begin{pmatrix}
y^{(i)}_{n+\frac{1}{2}}\\ y^{(i)}_{n+1}\\ y^{(i)}_{n+\frac{3}{2}}\\ y^{(i)}_{n+2}
\end{pmatrix}
+
\begin{pmatrix}
0 & 0 & 0 & \frac{9}{20}\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
f^{(i)}_{n-\frac{3}{2}}\\ f^{(i)}_{n-1}\\ f^{(i)}_{n-\frac{1}{2}}\\ f^{(i)}_{n}
\end{pmatrix}
+
\begin{pmatrix}
\varepsilon_{\frac{1}{2}}\\ \varepsilon_1\\ \varepsilon_{\frac{3}{2}}\\ \varepsilon_2
\end{pmatrix}. \quad (39)
$$

Tested Problems

To validate the efficiency of the methods developed, the following stiff IVPs are solved:

1. $y_1' = -20y_1 - 19y_2 \; y_1(0) = 2, \qquad 0 \le x \le 20,$
$y_2' = -19y_1 - 20y_2 \; y_1(0) = 0.$

Exact solutions: $y_1(x) = e^{-39x} + e^{-x}$
$y_2(x) = e^{-39x} - e^{-x}$

Eigen values: -1 and -39 **Source:** Musa (2015)

2. $y_1' = 198y_1 + 199y_2 y_1(0) = 1, \quad 0 \le x \le 10$
$y_2' = -398y_1 - 399y_2 y_2(0) = -1$.

Exact solution: $y_1(x) = e^{-x}$

$y_2(x) = -e^{-x}$

Eigen values: -1 and -200 **Source: Ibrahim (2007).**

3. $y' = 20y + 20\sin x + \cos x, \qquad y(0) = 1, \qquad 0 \le x \le 2,$
Exact Solution: $y(x) = \sin x + e^{-20x}$.Source: Abasi (2014)

4. $y' = -100(y - x) + 1, \qquad y(0) = 1, \qquad 0 \le x \le 10,$
Exact Solution: $y(x) = e^{-100x} + x$.Source: Abasi (2014)

Numerical Results

The numerical results for the test problems given in section 6 are tabulated. The problems are solved with 2OBBDF and 2ODISBBDF methods. The number of step taken to complete integration and maximum error for the different methods is presented and compared in the tables below. In addition, the graph of $Log_{10}(MAXE)$ against h for each problem is plotted (figure 2 - 5) in order to give the visual impact on the performance of the method.The notations used in the tables are listed below:

The following abbreviations are used in the tables:

2ODISBBDF =2-point super class BBDF with off-step points
2OBBDF =2-point block BDF method with off-step points of order 5
h =step size
NS=total number of steps
MAXE=maximum error
Time=computational time in seconds

Table 1: Numerical Results for Problems 1, 2, 3 and 4

	??	**Method**	**NS**	**MAXE**	**Time**
1.	98^{-2}	2OBBDF	1000	7.00088e-002	2.15117e-003
		2ODISBBDF	1000	3.81561e-002	1.73800e-001
	98^{-4}	2OBBDF	100000	2.84492e-003	2.06491e-001
		2ODISBBDF	100000	1.64714e-005	2.01139e+000
	98^{-6}	2OBBDF	10000000	2.87417e-005	6.00132e+001
		2ODISBBDF	10000000	1.70657e-009	1.19700e+002
2.	98^{-2}	2OBBDF	500	7.17251e-003	1.40432e-003
		2ODISBBDF	500	1.03577e-004	1.68700e-001
	98^{-4}	2OBBDF	50000	7.35564e-005	1.41352e-001
		2ODISBBDF	50000	1.12034e-008	1.13470e+000
	98^{-6}	2OBBDF	5000000	7.35775e-007	2.41100e+000
		2ODISBBDF	5000000	1.96752e-010	9.01800e+001
3.	98^{-2}	2OBBDF	100	8.05923e-002	5.90201e-004
		2ODISBBDF	100	1.86882e-002	1.18580e-001
	98^{-4}	2OBBDF	10000	1.46355e-003	2.01000e-002
		2ODISBBDF	10000	4.39784e-006	5.03090e-001
	98^{-6}	2OBBDF	100000	1.47126e-005	2.98923e+000
		2ODISBBDF	100000	4.48628e-010	3.76200e+001
4.	98^{-2}	2OBBDF	500	1.95750e-002	3.08300e-003
		2ODISBBDF	500	2.62911e-002	1.99800e-001
	98^{-4}	2OBBDF	50000	7.16455e-003	5.92900e-002
		2ODISBBDF	50000	1.03577e-004	1.38300e+000
	98^{-6}	2OBBDF	5000000	7.35564e-005	9.91000e+000
		2ODISBBDF	5000000	1.12034e-008	1.15600e+002

To give the visual impact on the performance of the method, the graphs of Graph of Log_{10} $(MAXE)$ against h for the problems tested are plotted. Given bellow are the graphs of the scaled maximum error arranged problem by problem.

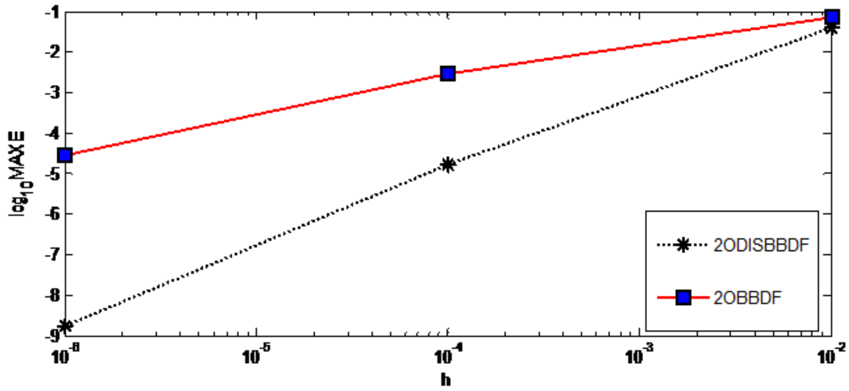


Figure 3: Graph of $Log_{10}(MAXE)$ against h for Problem 1

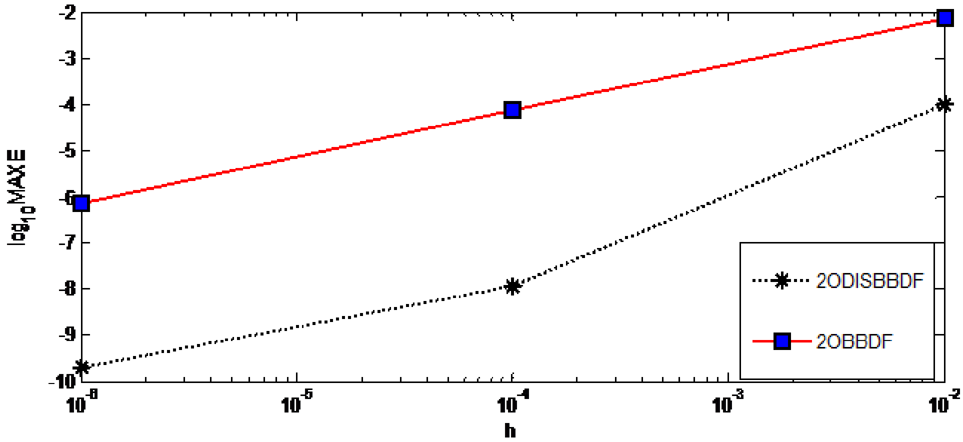


Figure 5: Graph of $Log_{10}(MAXE)$ against h for Problem 3

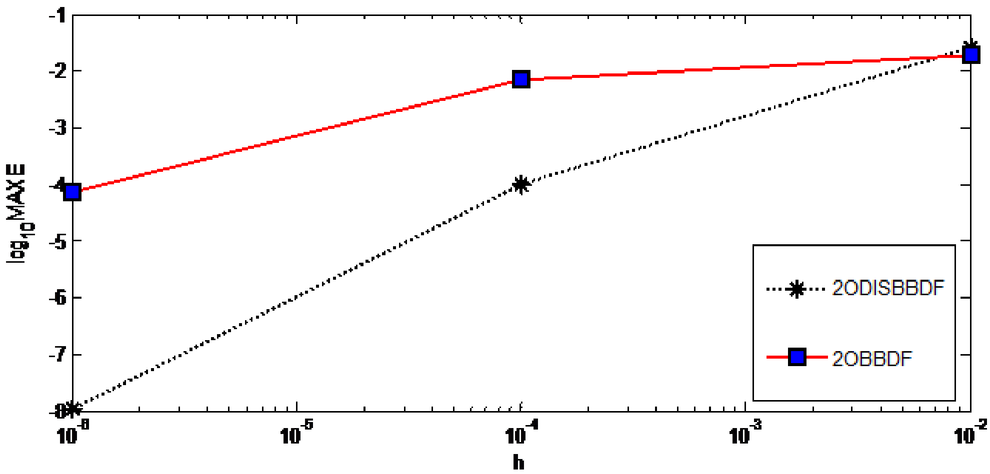


Figure 6: Graph of $Log_{10}(MAXE)$ against h for Problem 4

From the table 1above it can be seen that 2ODISBBDF method outperformed 2OBBDF method in terms of accuracy. The graphs also show that the scaled errors for the 2ODISBBDF method are smaller when compared with that in 2OBBDF method.

Conclusion and Recommendation

A new method of order 2 that is suitable for solving stiff initial value problems has been developed. The stability analysis has shown that the method is both zero and A-stable. Accuracy and the execution time of the derived method are compared with the existing 2-point block backward differentiation formula with off-step points (2OBBDF). This comparison shows that the new method outperformed the existing 2OBBDF method in terms of accuracy. The computation time for the new method is seen to be competitive. The graphs also show that the scaled errors for the 2ODISBBDF method are smaller when compared with that in 2OBBDF method.

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